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TANNAKA-KREIN DUALITY FOR COMPACT GROUPOIDS II, FOURIER TRANSFORM

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Abstract. In a series of papers, we have shown that from the representation theory of a compact groupoid one can reconstruct the groupoid using the procedure similar to the Tannaka-Krein duality for compact groups. In this part we study the Fourier and Fourier-Plancherel transforms and prove the Plancherel theorem for compact groupoids. We also study the central functions in the algebra of square integrable functions on the isotropy groups.

1. introduction

In a series of papers, we have generalized the Tannaka-Krein duality to compact groupoids. In [A1] we studied the representation theory of compact groupoids. In particular, we showed that irreducible representations have finite dimensional fibres. We also proved the Schur's lemma, Gelfand-Raikov theorem and Peter-Weyl theorem for compact groupoids. In this part we study the Fourier and Fourier-Plancherel transforms on compact groupoids. In section two we develop the theory of Fourier transforms on the Banach algebra bundle $L^1(\mathcal{G})$ of a compact groupoid $\mathcal{G}$. As in the group case, a parallel theory of Fourier-Plancherel transform on the Hilbert space bundle $L^2(\mathcal{G})$ is constructed. This provides a surjective isometric linear isomorphism from $L^2(\mathcal{G})$ to $L^2(\hat{\mathcal{G}})$, in an appropriate sense. Also the relation between $\hat{\mathcal{G}}$ and the conjugacy groupoid $\mathcal{G}^{\mathcal{G}}$ is studied. The results of this section are effectively used in [A2] to show that the natural homomorphism from $\mathcal{G}$ to its Tannaka groupoid $\mathcal{T}(\mathcal{G})$ is surjective. Section three considers the inverse Fourier and Fourier-Plancherel transforms. In this section we prove Plancherel theorem for compact groupoids. In section four we study the class functions and central elements in the algebras of functions on fibres of $\mathcal{G}$ and prove a diagonal version of the Plancherel theorem. All over this paper we assume that $\mathcal{G}$ is compact and the Haar system on $\mathcal{G}$ is normalized.

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2. Fourier transform

It follows from the Peter-Weyl theorem [A1, theorem 3.13] that, for $u, v \in X$, if $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$ then

$$f = \sum_{\pi \in \hat{\mathcal{G}}} \sum_{i=1}^{d_v^\pi} \sum_{j=1}^{d_u^\pi} c_{u,v,\pi}^{ij} \pi_{u,v}^{ij} ,$$

where

$$c_{u,v,\pi}^{ij} = d_u^\pi \int_{\mathcal{G}_u^v} f(x) \overline{\pi_{u,v}^{ij}(x)} d\lambda_u^v(x) \quad (1 \leq i \leq d_v^\pi,\, 1 \leq j \leq d_u^\pi).$$

This is a local version of the classical non commutative Fourier transform. As in the classical case, the main drawback is that it depends on the choice of the basis (which in turn gives the choice of the coefficient functions). The trick is similar to the classical case, that's to use the continuous decomposition using integrals. This is the content of the next definition. As usual, all the integrals are supposed to be on the support of the measure against which they are taken.

Definition 2.1. *Let $u, v \in X$ and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then the Fourier transform of f is $\mathfrak{F}_{u,v}(f) : \mathcal{R}ep(\mathcal{G}) \to \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ defined by*

$$\mathfrak{F}_{u,v}(f)(\pi) = \int f(x)\pi(x^{-1}) d\lambda_u^v(x).$$

To better understand this definition, let us go back to the group case for a moment. Let's start with a locally compact Abelian group G. Then the Pontryagin dual $\hat{G}$ of G is a locally compact Abelian group and for each $f \in L^1(G)$, its Fourier transform $\hat{f} \in C_0(\hat{G})$ is defined by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)} dx \quad (\chi \in \hat{G}).$$

The continuity of $\hat{f}$ is immediate and the fact that it vanishes at infinity is the so called *Riemann-Lebesgue lemma.* In the non Abelian compact case, one get a similar construction, namely, with an slight abuse of notation, for each $f \in L^1(G)$ one has $\hat{f} \in C_0(\hat{G}, \mathcal{B}(\mathcal{H}))$, where $\hat{G}$ is the set of (unitary equivalence classes of) irreducible representations of G endowed with the Fell topology, and $\mathcal{B}(\mathcal{H})$ is a bundle of C^*-algebras over $\hat{G}$ whose fiber at π is $\mathcal{B}(\mathcal{H}_\pi)$, and by $C_0(\hat{G}, \mathcal{B}(\mathcal{H}))$ we mean the set of all continuous sections which vanish at infinity. In the groupoid case, one has a similar interpretation. Locally each $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$ has its Fourier transform $\mathfrak{F}_{u,v}(f)$ in $C_0(\hat{\mathcal{G}}, \mathcal{B}_{u,v}(\mathcal{H}))$, where $\hat{\mathcal{G}}$ is the set of (unitary equivalence classes of) irreducible representations of $\mathcal{G}$ endowed again with the Fell topology, and $\mathcal{B}_{u,v}(\mathcal{H})$ is a bundle of C^*-algebras over $\hat{\mathcal{G}}$ whose fiber at π is $\mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, and $C_0(\hat{G}, \mathcal{B}_{u,v}(\mathcal{H}))$ is the set of all continuous sections vanishing at infinity. Globally we have an still more complicated interpretation. We have to look at $L^1(\mathcal{G})$ as

a bundle of Banach algebras over $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ whose fiber at (u,v) is $L^1(\mathcal{G}_u^v, \lambda_u^v)$, and then each $f \in L^1(\mathcal{G})$ has its Fourier transform $\mathfrak{F}(f)$ in $C_0(\hat{\mathcal{G}}, \mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ is a bundle of bundles of C^*-algebras over $\hat{\mathcal{G}}$ whose fiber at π is the bundle $\mathcal{B}(\mathcal{H}_\pi)$ over $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ whose fiber at (u,v) is $\mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, the space $C_0(\hat{G}, \mathcal{B}(\mathcal{H}))$ is the set of all continuous sections vanishing at infinity, and $\mathfrak{F}(f)(\pi)_{(u,v)} = \mathfrak{F}_{u,v}(f_{(u,v)})(\pi)$!!

Now let us discuss the properties of the Fourier transform. If we choose (possibly infinite) orthonormal bases for $\mathcal{H}_u^\pi$ and $\mathcal{H}_v^\pi$ and let each $\pi(x)$ be represented by the (possibly infinite) matrix with components $\pi_{u,v}^{ij}(x)$, then $\mathfrak{F}_{u,v}(f)$ is represented by the matrix with components $\mathfrak{F}_{u,v}(f)(\pi)^{ij} = \frac{1}{d_u^\pi} c_{u,v,\pi}^{ji}$. When $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$, summing up over all indices i,j, we get the following

Proposition 2.2. (Fourier inversion formula) *For each $u, v \in X$ and $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$,*

$$f = \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi Tr\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(.)\big),$$

where the sum converges in the L^2 norm and

$$\|f\|_2^2 = \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi Tr\big(\mathfrak{F}_{u,v}(f)(\pi)^* \mathfrak{F}_{u,v}(f)(\pi)\big).$$

□

We collect the properties of the Fourier transform in the following lemma. The proof is routine and is omitted.

Lemma 2.3. *Let $u, v \in X$, $a, b \in \mathbb{C}$, $x \in \mathcal{G}$, and $f, g \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then for each $\pi \in \mathcal{R}ep(\mathcal{G})$,*
$(i)\mathfrak{F}_{u,v}(af + bg) = a\mathfrak{F}_{u,v}(f) + b\mathfrak{F}_{u,v}(g)$,
$(ii)\mathfrak{F}_{u,v}(f * g)(\pi) = \mathfrak{F}_{u,v}(f)(\pi)\mathfrak{F}_{u,v}(g)(\pi)$,
$(iii)\mathfrak{F}_{u,v}(f^*)(\pi) = \mathfrak{F}_{u,v}(f)(\pi)^*$,
$(iv)\mathfrak{F}_{u,v}(\ell_x(f))(\pi) = \mathfrak{F}_{u,v}(f)(\pi)\pi(x^{-1})$, $\mathfrak{F}_{u,v}(r_x(f))(\pi) = \pi(x)\mathfrak{F}_{u,v}(f)(\pi) \quad (x \in \mathcal{G}_u^v)$. □

Corollary 2.4. *For $a, b \in \mathbb{C}$, $f, g \in L^1(\mathcal{G})$, and $\pi \in \hat{\mathcal{G}}$,*
$(i)\mathfrak{F}(af + bg) = a\mathfrak{F}(f) + b\mathfrak{F}(g)$,
$(ii)\mathfrak{F}(f * g)(\pi) = \mathfrak{F}(f)(\pi)\mathfrak{F}(g)(\pi)$,
$(iii)\mathfrak{F}(f^*)(\pi) = \mathfrak{F}(f)(\pi)^*$,

As in the group case there is yet another way of introducing the Fourier transform. For each finite dimensional continuous representation π of $\mathcal{G}$, let the *character* χ_π of π be the bundle of functions χ_π whose fiber at $u \in X$ is $\chi_u^\pi(x) = Tr(\pi(x)) \quad (x \in \mathcal{G}_u^u)$, where Tr is the trace of matrices. Note that one can not have these as functions defined on $\mathcal{G}_u^v$, since when $x \in \mathcal{G}_u^v$, $\pi(x)$ is not a square matrix in general. Also note that the values of the above character functions depend only on

the unitary equivalence class of π, as similar matrices have the same trace. Now if $\pi \in \hat{\mathcal{G}}$, $x \in \mathcal{G}_u^v$, and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then

$$Tr\big(\mathfrak{F}_{u,v}(f)(\pi)\pi(x)\big) = \int f(y)Tr(\pi(y^{-1}x))d\lambda_u^v(y) = f * \chi_u^\pi(x),$$

so it follows from Proposition 2.2 that

Corollary 2.5. *The map $P_{u,v}^\pi : L^2(\mathcal{G}_u^v, \lambda_u^v) \to \mathcal{E}_{u,v}^\pi, \quad f \mapsto d_u^\pi f * \chi_u^\pi$ is a surjective orthogonal projection and for each $f \in L^2(\mathcal{G}_u^v, \lambda_u^v)$ we have the decomposition*

$$f = \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi f * \chi_u^\pi,$$

which converges in the L^2 norm. □

Applying the above decomposition to the case where $u = v$ and $f = \chi_u^\pi$, we get

Corollary 2.6. *For each $u \in X$ and $\pi, \pi' \in \hat{\mathcal{G}}$,*

$$\chi_u^\pi * \chi_u^{\pi'} = \begin{cases} d_u^{\pi^{-1}} & \text{if } \pi \sim \pi', \\ 0 & \text{otherwise.} \end{cases}$$

□

3. inverse Fourier and Fourier-Palncherel transforms

Next we are aiming at the construction of the inverse Fourier transform. This is best understood if we start with a yet different interpretation of the local Fourier transform. It is clear from the definition that if $u, v \in X$, $\pi_1, \pi_2 \in \mathcal{R}ep(\mathcal{G})$, and $f \in L^1(\mathcal{G}_u^v, \lambda_u^v)$, then

$$\mathfrak{F}_{u,v}(f)(\pi_1 \oplus \pi_2) = \mathfrak{F}_{u,v}(f)(\pi_1) \oplus \mathfrak{F}_{u,v}(f)(\pi_2),$$

and the same is true for any number (even infinite) of continuous representations, so it follows from Theorem 2.16 in [A1] that $\mathfrak{F}_{u,v}(f)$ is uniquely characterized by its values on $\hat{\mathcal{G}}$, namely we can regard

$$\mathfrak{F}_{u,v} : L^1(\mathcal{G}_u^v, \lambda_u^v) \to \prod_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi).$$

Now consider the C^*-algebra ℓ^∞-direct sum $\sum_{\pi \in \hat{\mathcal{G}}} \bigoplus \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$. The domain of our inverse Fourier transform then would be the algebraic sum ℓ^∞-direct sum $\sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$, consisting of those elements of this C^*-algebra with only finitely many nonzero components.

Definition 3.1. *Let $u, v \in X$. The inverse Fourier transform*

$$\mathfrak{F}_{u,v}^{-1} : \sum_{\pi \in \hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi) \to C(\mathcal{G}_u^v)$$

is defined by

$$\mathfrak{F}_{u,v}^{-1}(g)(x) = \sum_{\pi \in \hat{\mathcal{G}}} d_u^{\pi} Tr\big(g(\pi)\pi(x^{-1})\big) \quad (x \in \mathcal{G}_u^v).$$

To show that this is indeed the inverse map of the (local) Fourier transform we need some orthogonality relations. They are a version of the Schur's orthogonality relations [A1, theorem 3.6] and proved similarly, so We only give a sketch of the proof.

Proposition 3.2. (Orthogonality relations) *Let $\tau, \rho \in \hat{\mathcal{G}}$, $u, v \in X$, $T \in \mathcal{B}(\mathcal{H}_\tau)$, $S \in \mathcal{B}(\mathcal{H}_\rho)$, $\xi \in \mathcal{H}_\tau$, $\eta \in \mathcal{H}_\rho$, and $A \in Mor(\tau, \rho)$, then*

$$(i) \quad \int \tau(x) A_{s(x)} \rho(x^{-1}) d\lambda_u(x) = \begin{cases} \frac{Tr(A_u)}{d_u^\tau} id_{\mathcal{H}_u^\tau} & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}, \end{cases}$$

$$(ii) \quad \int \tau(x) \xi_{s(x)} \otimes \rho(x^{-1}) \eta_{r(x)} d\lambda_u(x) = \begin{cases} \frac{\eta_u \otimes \xi_u}{d_u^\tau} & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}, \end{cases}$$

$$(iii) \quad \int Tr(T_{r(x)}\tau(x)) Tr(S_{s(x)}\rho(x^{-1}) d\lambda_u(x) = \begin{cases} \frac{Tr(T_u S_u)}{d_u^\tau} & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}, \end{cases}$$

$$(iv) \quad \int Tr(T_{r(x)}\tau(x)) \overline{Tr(S_{r(x)}\rho(x)} d\lambda_u(x) = \begin{cases} \frac{Tr(T_u S_u^*)}{d_u^\tau} & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}, \end{cases}$$

$$(v) \quad \int Tr(T_{r(x)}\tau(x)) \rho(x^{-1}) d\lambda_u(x) = \begin{cases} \frac{1}{d_u^\tau} T_u & \textit{if } \tau = \rho, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof (i) As in [A1, lemma 3.4], the left hand side defines a bundle of operators in $Mor(\tau, \rho)$, so by Schur's lemma [A1, theorem 2.14] it is $c.id_{\mathcal{H}_u^\tau}$, if $\tau = \rho$, and 0, otherwise. Now

$$Tr\Big(\int \tau(x) A_{s(x)} \tau(x^{-1}) d\lambda_u(x)\Big) = Tr(A_u),$$

where as $Tr\big(c.id_{\mathcal{H}_u^\tau}\big) = c d_u^\tau$, so c is what it should be.

(ii) Take any $\phi, \psi \in \mathcal{H}_\tau^*$ and apply (i) to A defined by $A_u(\zeta_u) = \phi_u(\zeta_u)\xi_u \quad (u \in X)$ and then calculate both sides of the resulting operator equation at η_u to get

$$\int \tau(x) \xi_{s(x)} \phi(\rho(x^{-1}\eta_u) d\lambda^u(x) = \begin{cases} \frac{\phi_u(\xi_u)}{d_u^\tau} \eta_u & \text{if } \tau = \rho, \\ 0 & \text{otherwise}. \end{cases}$$

The result now follows if we apply ψ_u to both sides of the above equality and use the fact that $\phi_u(\xi_u)\psi_u(\eta_u) = (\psi \otimes \phi)_u(\eta_u \otimes \xi_u)$.

$(iii), (iv)$ Note that all the involved Hilbert spaces are finite dimensional [A1, theorem 2.16]. In particular, rank one operators generate all operators on these spaces. Also the required relation is linear in T and S. Hence we may assume that T and S have rank one fibers, say $T = \phi(.)\xi, S = \psi(.)\eta$, where ϕ, ψ are as above. Now applying $(\phi \otimes \psi)_u$ to both sides of (ii), we get (iii). The proof of (iv) is similar.

(v) Let L and R be the left and right hand sides of (v), respectively. We need only to show that $Tr((L-R)S) = 0$, for each $S \in \mathcal{B}(\mathcal{H}_\rho)$. But $Tr(LS)$ is clearly the right hand side of (iii), which is in turn equal to $Tr(RS)$. □

Now we are ready to prove the properties of the local inverse Fourier transform. But let us first introduce the natural inner products on its domain and range. For $f, g \in C(\mathcal{G}_u^v)$ and $h, k \in \sum_{\pi\in\hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ put

$$< f, g >= \int \bar{f}.g d\lambda_u^v,$$

and

$$< h, k >= \sum_{\pi\in\hat{\mathcal{G}}} d_u^\pi Tr(h^*(\pi)k(\pi)),$$

where the right hand side is a finite sum as h and k are of finite support. Also note that if $\varepsilon_u : C(\mathcal{G}_u^u) \to \mathbb{C}$ is defined by $\varepsilon_u(f) = f(u)$, then for each $f, g \in C(\mathcal{G}_u^v)$, we have $f^* * g \in C(\mathcal{G}_u^u)$ and $< f, g >= \varepsilon_u(f^* * g)$.

Proposition 3.3. *For each $u, v \in X$ and $h, k \in \sum_{\pi\in\hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ we have*

$(i)\mathfrak{F}_{u,v}\mathfrak{F}_{u,v}^{-1}(h) = h,$

$(ii)\mathfrak{F}_{u,v}^{-1}(hk) = \mathfrak{F}_{u,v}^{-1}(h) * \mathfrak{F}_{u,v}^{-1}(k),$

$(iii)\mathfrak{F}_{u,v}^{-1}(h^*) = (\mathfrak{F}_{u,v}^{-1}(h))^*,$

$(iv) < \mathfrak{F}_{u,v}^{-1}(h), \mathfrak{F}_{u,v}^{-1}(k) >=< h, k >.$

Proof (i) By (v) of above proposition, for each $\tau \in \hat{\mathcal{G}}$,

$$\mathfrak{F}_{u,v}\mathfrak{F}_{u,v}^{-1}(h)(\tau) = \sum_{\pi\in\hat{\mathcal{G}}} \int d_u^\pi Tr(h(\pi)\pi(x^{-1}))\tau(x) d\lambda_u^v(x) = h(\tau).$$

(ii) By (iii) of above proposition, for each $x \in \mathcal{G}_u^v$,

$$\begin{aligned}
(\mathfrak{F}_{u,v}^{-1}(h) * \mathfrak{F}_{u,v}^{-1}(k))(x) &= \int \mathfrak{F}_{u,v}^{-1}(h)(xy^{-1})\mathfrak{F}_{u,v}^{-1}(k)(y)d\lambda_u^v(y) \\
&= \sum_{\tau,\rho\in\hat{\mathcal{G}}} \int d_u^\tau d_u^\rho Tr\big(h(\tau)\tau(yx^{-1})\big)Tr\big(k(\rho)\rho(y^{-1})\big)d\lambda_u^v(x) \\
&= \sum_{\tau,\rho\in\hat{\mathcal{G}}} d_u^\tau d_u^\rho \int Tr\big(h(\tau)\tau(x^{-1})\tau(y)\big)Tr\big(k(\rho)\rho(y^{-1})\big)d\lambda_u^v(x) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(h(\tau)\tau(x^{-1})k(\tau)\big) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(h(\tau)k(\tau)\tau(x^{-1})\big) \\
&= \mathfrak{F}_{u,v}^{-1}(hk)(x).
\end{aligned}$$

(iii) For each $x \in \mathcal{G}_u^v$

$$\begin{aligned}
\mathfrak{F}_{u,v}^{-1}(h^*)(x) &= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(h^*(\tau)\tau(x^{-1})\big) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(\bar{h}((\check{\tau})\bar{})\tau(x^{-1})\big) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(\bar{h}(\tau)\bar{\bar{\tau}}(x^{-1})\big) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(\bar{h}(\tau)\bar{\tau}(x)\big),
\end{aligned}$$

where as

$$\begin{aligned}
(\mathfrak{F}_{u,v}^{-1}(h))^*(x) &= \overline{\mathfrak{F}_{u,v}^{-1}(h)(x^{-1})} \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau \overline{Tr\big(h(\tau)\tau(x)\big)} \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(\bar{h}(\tau)\bar{\tau}(x)\big).
\end{aligned}$$

(iv) By above observation about ε_u,

$$\begin{aligned}
< \mathfrak{F}_{u,v}^{-1}(h), \mathfrak{F}_{u,v}^{-1}(k) > &= \varepsilon_u((\mathfrak{F}_{u,v}^{-1})^* * \mathfrak{F}_{u,v}^{-1}(h)) = \varepsilon_u(\mathfrak{F}_{u,v}^{-1}(h^*k)) = \mathfrak{F}_{u,v}^{-1}(h^*k)(u) \\
&= \sum_{\tau\in\hat{\mathcal{G}}} d_u^\tau Tr\big(h^*(\tau)k(\tau)\big) =< h, k > .
\end{aligned}$$

□

Next we define a norm on the domain of the inverse Fourier transform in order to get a Plancherel type theorem. For $h \in \sum_{\pi\in\hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}_v^\pi, \mathcal{H}_u^\pi)$ we put $\|h\|_2 =< h, h >^{\frac{1}{2}}$. This is the natural norm on the algebraic direct

sum, when one endows each component $\mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$ with the pre-Hilbert space structure given by $< T, S >= \left(d^\pi_u Tr(S^*T)\right)^{\frac{1}{2}}$. We denote the completion of $\sum_{\pi\in\hat{\mathcal{G}}} \mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$ with respect to this norm by $\mathcal{L}^2_{u,v}(\mathcal{G})$. We deliberately used the curly $\mathcal{L}$ for this space to distinguish it from its counterpart which is defined later.

Theorem 3.4. (Plancherel Theorem) *For each $u, v \in X$, $\mathfrak{F}_{u,v}$ extends (uniquely) to a continuous surjective linear isometry $\mathfrak{F}_{u,v}$: $L^2(\mathcal{G}^v_u, \lambda^v_u) \to \mathcal{L}^2_{u,v}(\mathcal{G})$.*

Proof By Proposition 3.4 (iv), $\mathfrak{F}^{-1}_{u,v} : \mathcal{L}^2_{u,v}(\mathcal{G}) \to L^2(\mathcal{G}^v_u, \lambda^v_u)$ is an isometric embedding. It is also surjective, since $Im(\mathfrak{F}^{-1}_{u,v})$ is complete and so closed, and also it clearly includes $\mathcal{E}_{u,v}$ which is dense in $L^2(\mathcal{G}^v_u, \lambda^v_u)$. □

The above map is called the (local)*Fourier-Plancherel transform.* Now for each $u, v \in X$, $\mathcal{G}^u_u$ and $\mathcal{G}^v_v$ act on both $L^2(\mathcal{G}^v_u, \lambda^v_u)$ and $\mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$, from right and left respectively, via

$$(f.x)(y) = f(yx),\ A.x = \pi(x)A \quad (x \in \mathcal{G}^u_u, y \in \mathcal{G}^v_u),$$

and

$$(x.f)(y) = f(x^{-1}y),\ x.A = A\pi(x) \quad (x \in \mathcal{G}^v_v, y \in \mathcal{G}^v_u),$$

for each $f \in L^2(\mathcal{G}^v_u, \lambda^v_u), A \in \mathcal{B}(\mathcal{H}^\pi_v, \mathcal{H}^\pi_u)$.

It is easy to see that the Fourier-Plancherel transform respects these actions, namely

Lemma 3.5. *For each $u, v \in X$, $x \in \mathcal{G}^v_v$, $y \in \mathcal{G}^u_u$, and $f \in L^2(\mathcal{G}^v_u, \lambda^v_u)$,*

$$\mathfrak{F}_{u,v}(x.f) = x.\mathfrak{F}_{u,v}(f),\ \mathfrak{F}_{u,v}(f.y) = x.\mathfrak{F}_{u,v}(f).$$

□

Before we end this section, let us show that how one can use characters of representations in $\hat{\mathcal{G}}$ and the orthogonality relations of the beginning of this section to prove statements about subsets of $\mathcal{R}ep(\mathcal{G})$.

Lemma 3.6. *For each $u \in X$ and $\pi \in \hat{\mathcal{G}}$, $\chi^\pi_u \in \mathcal{E}_{u,u}$.*

Proof Let $\{e^i_u\}_{1\le i\le d^\pi_u}$ be a basis for $\mathcal{H}^\pi_u$, then

$$\chi^\pi_u = Tr(\pi(.)) = \sum_{i=1}^{d^\pi_u} < \pi(.)e^\pi_u, e^\pi_u >,$$

so the result follows from Proposition 3.2 of [A1]. □

Definition 3.7. *A subset Σ of $\mathcal{R}ep(\mathcal{G})$ is called closed if it contains*

(i) π_1 if π_1 is unitary equivalent to some $\pi_2 \in \Sigma$,
(ii) π_1 if π_1 is weakly contained in some $\pi_2 \in \Sigma$,
(iii) $\pi_1 \oplus \pi_2$ if π_1, π_2 are in Σ,
(iv) $\pi_1 \otimes \pi_2$ if π_1, π_2 are in Σ,

(v) $\bar{\pi}_1$ if π_1 is in Σ,
(vi) the trivial representation tr.

Proposition 3.8. *If $\Sigma \subseteq \mathcal{R}ep(\mathcal{G})$ is closed and separates the points of $\mathcal{G}$, then $\Sigma = \mathcal{R}ep(\mathcal{G})$.*

Proof If not, by condition (iii) of the definition of closedness and Theorem 2.16 of [A1], there is $\tau \in \hat{\mathcal{G}}$ which is not in Σ. Let $\mathcal{E}^{\Sigma}_{u,v} = \cup_{\pi\in\Sigma}\mathcal{E}^{\pi}_{u,v}$, for $u, v \in X$. By Proposition 3.2, elements of each $\mathcal{E}^{\tau}_{u,v}$ is orthogonal to $\mathcal{E}^{\Sigma}_{u,v}$. In particular, by above lemma, $\chi^{\tau}_u \in (\mathcal{E}^{\Sigma}_{u,v})^{\perp}$. But by conditions (iii)-(v) of the definition of closedness, $\mathcal{E}^{\Sigma}_{u,v}$ is a subalgebra of $C(\mathcal{G})$ which is closed under conjugation, and by condition (vi), it contains the constants, and finally by assumption,it separates the points of $\mathcal{G}$. Hence, by Stone-Weierstrass Theorem, $\mathcal{E}^{\Sigma}_{u,v}$ is dense in $C(\mathcal{G})$. Therefore χ^{τ}_u is orthogonal to $C(\mathcal{G})$ and so it is zero, which is a contradiction. □

4. Class functions and central elements

We characterize central elements in $C(\mathcal{G}^v_u)$ and $L^2(\mathcal{G}^v_u, \lambda^v_u)$, with respect to the convolution product.

Definition 4.1. *A function $f \in C(\mathcal{G})$ is called a class function if it is constant on "conjugacy classes" of $\mathcal{G}$, that is*

$$f(x^{-1}yx) = f(y) \quad (u \in X, x \in \mathcal{G}^u, y \in \mathcal{G}^u_u).$$

We denote the set of all class functions on $\mathcal{G}$ by $\mathfrak{C}C(\mathcal{G})$.

When we talk about $C(\mathcal{G})$ as an algebra we always consider it with the pointwise multiplication. However, it is clear that $C(\mathcal{G})$ is also an algebra with respect the convolution. This later algebra is non commutative in general, and to distinguish it from $C(\mathcal{G})$ we will denote it with $(C(\mathcal{G}), *)$. Also we denote the center of and algebra $\mathcal{A}$ with $\mathcal{Z}(\mathcal{A})$.

Lemma 4.2. $\mathfrak{C}C(\mathcal{G}) = \mathcal{Z}(C(\mathcal{G}), *)$.

Proof If $f \in \mathfrak{C}C(\mathcal{G})$ and $g \in C(\mathcal{G})$, then for each $x \in \mathcal{G}$,

$$\begin{aligned}
(f * g)(x) &= \int f(xy^{-1})g(y)d\lambda_{s(x)}(y) \\
&= \int f(xy^{-1}x^{-1})g(xy)d\lambda^{s(x)}(y) \\
&= \int f(y^{-1})g(xy)d\lambda^{s(x)}(y) \\
&= \int f(y)g(xy^{-1})d\lambda_{s(x)}(y). \\
&= (g * f)(x)
\end{aligned}$$

Conversely if $f \in \mathcal{Z}(C(\mathcal{G}), *)$, then for each $u \in X$, $x \in \mathcal{G}^u$, and $g \in C(\mathcal{G})$,

$$\int (f(xyx^{-1}) - f(y))g(xy^{-1})d\lambda_u^u(y) = 0,$$

so by continuity of f, $f(xyx^{-1}) - f(y) = 0$ for each $y \in \mathcal{G}_u^u$. $\square$

Recall that each $f \in L^1(\mathcal{G})$ has a global Fourier transform $\mathfrak{F}(f)$ such that for each $\pi \in \hat{\mathcal{G}}$, $\mathfrak{F}(f)(\pi)$ is fibred over $X \times X$. For some technical reasons, sometimes we need to consider everything to be fibred over X (rather than $X \times X$. This is naturally done by considering a diagonal version of this global Fourier transform, namely $\mathfrak{F}(f)(\pi)$ is fibred over X and its fiber at u is $\mathfrak{F}_{u,u}(f_{(u,u)})$. To distinguish these two, we denote the diagonal version of the global Fourier transform by $\mathfrak{DF}$.

Now each $f \in C(\mathcal{G})$ could be considered as a an element in $L^1(\mathcal{G})$ whose fiber at (u, v) is the restriction of f to $\mathcal{G}_u^v$.

Lemma 4.3. *If $f \in \mathfrak{C}C(\mathcal{G})$ then for each $\pi \in \hat{\mathcal{G}}$, $\mathfrak{DF}(f)(\pi) \in Mor(\pi, \pi)$.*

Proof Let $\pi \in \hat{\mathcal{G}}$. By Lemma 4.2 and definition of $\mathfrak{C}C(\mathcal{G})$, for each $u \in X$ and $x \in \mathcal{G}_u^u$ we have

$$x.\mathfrak{F}_{u,u}(f)(\pi) = \mathfrak{F}_{u,u}(x.f)(\pi) = \mathfrak{F}_{u,u}(f.x)(\pi) = \mathfrak{F}_{u,u}(f)(\pi).x.$$

$\square$

Up to now our Fourier transform was an operator valued map. As in the group case, we can use the characters of irreducible representations to define a complex valued version of the Fourier transform.

Definition 4.4. *Let $\mathcal{G}$ be a compact groupoid and $f \in L^1(\mathcal{G})$. Consider $L^1(\mathcal{G})$ as a bundle over X whose fiber at $u \in X$ is $L^1(\mathcal{G}_u, \lambda_u)$. Define $\hat{f}$ on $\hat{\mathcal{G}}$ fibrewise by*

$$\hat{f}_u(\pi) = \frac{1}{d_u^\pi}\int f_u(x)\chi_u^\pi(x^{-1})d\lambda_u^u(x) = \frac{1}{d_u^\pi}\int f_u(x)Tr(\pi(x^{-1}))d\lambda_u^u(x) \quad (u \in X).$$

We call $\hat{f}$ the diagonal Fourier transform of f (the terminology is justified with the next proposition).

Because of the restriction imposed by the trace, we had to look at the $L^1(\mathcal{G})$ as a bundle over X (rather than $X \times X$). As one might guess, this makes our new version of the Fourier transform compatible with the diagonal form of the global Fourier transform.

Proposition 4.5. *For $f \in \mathfrak{C}C(\mathcal{G})$, $\mathfrak{DF}(f)(\pi) = \hat{f}(\pi)id_{\mathcal{H}_\pi} \quad (\pi \in \hat{\mathcal{G}})$.*

Proof First note that the above equality means that for each $u \in X$, $\mathfrak{F}_{u,u}(f)(\pi) = \hat{f}_u(\pi)id_{\mathcal{H}_u^\pi}$. It follows from above lemma and Schur's lemma [A1, theorem 2.14], that $\mathfrak{DF}(f)(\pi) = c_\pi id_{\mathcal{H}_\pi}$, for some bundle

of constants $c_\pi = \{c_u^\pi\}$. But then for $u \in X$,

$$c_u^\pi = \frac{1}{d_u^\pi} Tr\big(\mathfrak{F}_{u,u}(f)(\pi)\big) = \frac{1}{d_u^\pi} Tr\Big(\int f_u(x)\pi(x^{-1})d\lambda_u^u(x)\Big) = \hat{f}_u(\pi).$$

□

The diagonal Fourier transform has the same properties as the global Fourier transform. The proof is routine.

Proposition 4.6. *For $a, b \in \mathbb{C}$, $f, g \in L^1(\mathcal{G})$, and $\pi \in \hat{\mathcal{G}}$,*
$(i)(af + bg)\hat{} = a\hat{f} + b\hat{g}$,
$(ii)(f * g)\hat{}(\pi) = \hat{f}(\pi)\hat{g}(\pi)$,
$(iii)(f^*)\hat{}(\pi) = \hat{f}(\pi)^*$. □

Remark 4.7. *In above proposition, if $f, g \in \mathfrak{C}C(\mathcal{G})$, one can easily check that $af + bg, f * g, f^* \in \mathfrak{C}C(\mathcal{G})$. In this case the above relations follow from Lemmas 2.3 and 4.2.*

Next we turn into the inverse of the diagonal Fourier transform. Here $\hat{\mathcal{G}}$ is endowed with the discrete topology, so for instance $C_c(\hat{\mathcal{G}})$ simply means all complex valued functions on $\hat{\mathcal{G}}$ with finite support [consider it as a bundle with fiber at u to be ?]. Also we consider $C(\mathcal{G})$ as a bundle over X whose fiber at $u \in X$ is $C(\mathcal{G}_u^u)$.

Definition 4.8. *Let $\mathcal{G}$ be a compact groupoid. The inverse diagonal Fourier transform from $C_c(\hat{\mathcal{G}})$ to $C(\mathcal{G})$ is defined by $g \mapsto \check{g}$, where*

$$\check{g}(x) = \begin{cases} \sum_{\pi\in\hat{\mathcal{G}}} d_u^\pi g(\pi)\overline{\chi_u^\pi(x)} & \text{if } x \in \mathcal{G}_u^u \text{ for some } u \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We have already used the notation $\check{g}$ with a different meaning, namely for a function g on $\mathcal{G}$, $\check{g}(x) = g(x^{-1})$. However, since this notation is now used only for functions on $\hat{\mathcal{G}}$, there is no fear of confusion.

Lemma 4.9. *If $g \in C_c(\hat{\mathcal{G}})$ then $\check{g} \in \mathfrak{C}(\mathcal{G})$. In particular, if $\check{g}$ is continuous then $\check{g} \in \mathfrak{C}C(\mathcal{G})$. This is the case, for instance when $X = \mathcal{G}^{(0)}$ is discrete in the relative topology of $\mathcal{G}$ (when $\mathcal{G}$ is Hausdorff, this means that X is finite).*

Proof First let us prove the last statement. If $x_\alpha \to x$ in $\mathcal{G}$, then by continuity of the source and range maps, $s(x_\alpha) \to u = s(x)$ and $r(x_\alpha) \to v = r(x)$. If X is discrete, eventually $s(x_\alpha) = u$ and $r(x_\alpha) = v$. If $u \neq v$, eventually $\check{g}(x_\alpha) = \check{g}(x) = 0$, otherwise eventually $\check{g}(x_\alpha) = \check{g}_u(x_\alpha) \to \check{g}_u(x) = \check{g}(x)$, by the fact that the character χ_u^π of each $\pi \in \hat{\mathcal{G}}$ is continuous.

If $g \in C_c(\hat{\mathcal{G}})$, then for each $f \in C(\mathcal{G})$ and $x \in \mathcal{G}$ with $u = s(x), v = r(x)$ we have

$$
\begin{aligned}
(\check{g} * f)(x) &= \int \check{g}(xy^{-1}) f(y) d\lambda_{s(x)}(y) \\
&= \int \check{g}(xy^{-1}) f(y) d\lambda_u^v(y) \\
&= \sum_{\pi \in \hat{\mathcal{G}}} \int d_u^\pi g(\pi) \overline{\chi_v^\pi(xy^{-1})} f(y) d\lambda_u^v(y) \\
&= \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi g(\pi) \int \overline{\chi_v^\pi(xyx^{-1})} f(xy^{-1}) d\lambda_u^u(y) \\
&= \sum_{\pi \in \hat{\mathcal{G}}} d_u^\pi g(\pi) \int \overline{\chi_u^\pi(y)} f(xy^{-1}) d\lambda_u^u(y) \\
&= \int \check{g}(y) f(xy^{-1}) d\lambda_u^u(y) \\
&= \int \check{g}(y) f(xy^{-1}) d\lambda_{s(x)}(y) \\
&= (f * \check{g})(x).
\end{aligned}
$$

□

The diagonal inverse Fourier transform satisfies the same properties as the global inverse Fourier transform. We define natural inner product on $C_c(\hat{\mathcal{G}})$ by

$$< g, h >= \sum_{\pi \in \hat{\mathcal{G}}} d_u^{\pi 2} \bar{g}(\pi) h(\pi) \quad (g, h \in C_c(\hat{\mathcal{G}})).$$

Proposition 4.10. *For each $g, h \in C_c(\hat{\mathcal{G}})$,*
$(i)(\check{g})\check{} = g$,
$(ii)(gh)\check{} = \check{g} * \check{h}$,
$(iii)(g^*)\check{} = (\check{g})^*$,
$(iv) < \check{g}, \check{h} >=< g, h >$. □

Next we characterize the central elements in $L^2(\mathcal{G})$.

Definition 4.11. *Consider $\hat{\mathcal{G}}$ with the discrete topology. The spectral measure d on $\hat{\mathcal{G}}$ is defined by*

$$d(\{\pi\}) = d_u^{\pi 2} \quad (\pi \in \hat{\mathcal{G}}).$$

Definition 4.12. *Let*

$$\mathcal{G}' = \{x \in \mathcal{G} : s(x) = r(x)\}$$

be the isotropy bundle of $\mathcal{G}$. For each $x \in \mathcal{G}'$, let the conjugacy class of x be

$$\dot{x} = \{y^{-1}xy : y \in \mathcal{G}^{s(x)}\},$$

and

$$\mathcal{G}'^{\mathcal{G}} = \{\dot{x} : x \in \mathcal{G}'\}.$$

Let $q : \mathcal{G}' \to \mathcal{G}'^{\mathcal{G}}$ be the canonical projection, $x \mapsto \dot{x}$. It is clear that $\mathcal{G}'$ is a closed (and so compact) subgroupoid of $\mathcal{G}$. Also $\mathcal{G}'^{(0)} = \mathcal{G}^{(0)} = X$. The Haar system $\{\lambda_u, \lambda^u\}_{u\in X}$ of $\mathcal{G}$ restricted to $\mathcal{G}'$ is a Haar system for $\mathcal{G}'$, which in turn transfers along q to a Haar system on $\mathcal{G}'^{\mathcal{G}}$, which we denote by $\{\dot{\lambda}_u, \dot{\lambda}^u\}_{u\in X}$.

Definition 4.13. *Consider $L^2(\mathcal{G})$ as a bundle of Hilbert spaces over X, whose fiber at $u \in X$ is $L^2(\mathcal{G}_u, \lambda_u)$. A function bundle $f = \{f_u\}_{u\in X} \in L^2(\mathcal{G})$ is called central if for each $u \in X$,*

$$f_u(x^{-1}yx) = f_u(y) \quad (x \in \mathcal{G}^u, y \in \mathcal{G}^u_u).$$

We denote the set of all central elements of $L^2(\mathcal{G})$ by $\mathfrak{C}L^2(\mathcal{G})$.

Lemma 4.14. $L^2(\mathcal{G}'^{\mathcal{G}}) \simeq \mathcal{Z}(L^2(\mathcal{G}')) = \mathfrak{C}L^2(\mathcal{G}')$.

Proof The second equality could be proved as in Lemma 4.2. For the first, let $f \in \mathfrak{C}L^2(\mathcal{G})$, then $\dot{f}$ defined by $\dot{f}_u(\dot{x}) = f_u(x) \quad (u \in X, x \in \mathcal{G}')$ is well defined. Also for each $u \in X$,

$$\|\dot{f}_u\|_2^2 = \int |\dot{f}_u(\dot{x})|^2 d\dot{\lambda}_u(\dot{x}) = \int |f_u(x)|^2 d\lambda_u(x) = \|f_u\|_2^2 < \infty.$$

Conversely if $g \in L^2(\mathcal{G}'^{\mathcal{G}})$, then f defined by $f_u(x) = g_u(\dot{x}) \quad (u \in X, x \in \mathcal{G}')$ is clearly in $\mathfrak{C}L^2(\mathcal{G}')$ and fibrewise has the same L^2 norm. □

All these lead to an alternative version of the Plancherel theorem, which could be proved similarly by restricting to central parts.

Theorem 4.15. (Palncherel Theorem, diagonal version) *The diagonal inverse Fourier transform* $\check{}: C_c(\hat{\mathcal{G}}) \to \mathfrak{C}C(\mathcal{G})$ *extends (uniquely) to a bijective linear isometry* $\check{} : L^2(\hat{\mathcal{G}}) \to L^2(\mathcal{G}'^{\mathcal{G}})$ *with inverse* $\hat{} : L^2(\mathcal{G}'^{\mathcal{G}}) \to L^2(\hat{\mathcal{G}})$. □

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